

## OLS/MLR Analytics II – Collinearity & Co.

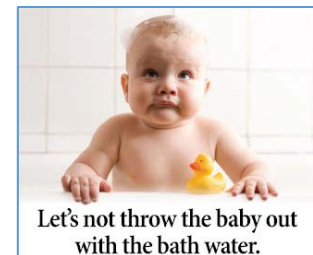
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### Introduction: (Multi)Collinearity

In moving from SLR and MLR models, you face the possibility of having multiple explanatory variables. As it turns out, how those variables move together (or not) ends up mattering in MLR analysis... a lot. We call that moving together *multicollinearity*, or *collinearity*, for short.<sup>1</sup>

Multicollinearity can impact a number of aspects of MLR analysis:

- **Omitted Variable Bias/Impact (Endogeneity):** It plays a role in driving omitted variable impact/bias (endogeneity) ... recall the previous discussion.
- **SRF interpretation of MLR coefficients:** It can wreak havoc with the *ceteris paribus* interpretation of MLR coefficients... Does it really make sense (when interpreting coefficients) to hold other things constant when things are moving together closely?
- **Standard Errors:** It impacts estimated standard errors and the precision with which parameters have been estimated. Greater collinearity leads to higher standard errors (and smaller t-stats) and less precision in estimation.<sup>2</sup>
- **Wacky results... But don't toss the baby!** And perhaps most insidiously, it can lead to wacky estimated coefficients, which in turn could very well lead the researcher astray... to focus on less important explanatory factors. You may be tempted to throw out all of the babies... and be left with no babies or bath water.
- **Explanatory power:** Less collinear potential explanatory variable candidates plausibly offer more *independent* explanatory power to the RHS, which could make them more attractive candidates for inclusion in a MLR analysis... so long, of course as they are also correlated with the dependent variable.



<sup>1</sup> Some like to distinguish between the two terms.... I don't.

<sup>2</sup> The topic of standard errors (and precision of estimation) is not covered here... but will come later with the discussion of statistical inference.

## OLS/MLR Analytics II: *Collinearity & Co.*

### The Collinearity Regressions

Consider a simple box office revenue forecasting MLR model in which total gross revenue, *rtotgross*, is regressed on weekly revenues in weeks #1 and #2.

The two collinearity regressions associated with this MLR model are the SLR regressions in which each of the RHS variables has been separately regressed on the other/remaining RHS variable.

Here's a summary of the results from the full MLR model (Model (1)), and the two associated collinearity regressions (Models (2) and (3)):

	<u>MLR Model</u>	<u>Collinearity Regressions</u>	
	(1)	(2)	(3)
	<u>rtotgross</u>	<u>wk1</u>	<u>wk2</u>
wk1	-0.0120 (-0.46)		0.522*** (249.91)
wk2	4.536*** (95.87)	1.673*** (249.91)	
_cons	0.401 (1.74)	-0.178 (-1.95)	0.889*** (17.75)
N	9114	9114	9114
R-sq	0.887	<u>0.873</u>	<u>0.873</u>

t statistics in parentheses  
\* p<0.05, \*\* p<0.01, \*\*\* p<0.001

### Multicollinearity with two RHS Variables: *Correlation and R-squared*

#### 1. *Correlation*

If there are just two RHS variables in the MLR model, then an obvious measure of collinearity would be the correlation of the two variables, which captures the extent to which the two variables are moving together in a linear fashion.

Here's the correlation between the two RHS variables in Model (1) above:

```
. corr wk1 wk2
(obs=9,114)

-----+-----
      |          wk1          wk2
-----+-----
wk1 |          1.0000
wk2 |          0.9342          1.0000
```

Not surprisingly, wk1 and wk2 revenues are highly correlated (0.9342).

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### 2. *Coefficient of Determination - $R^2$*

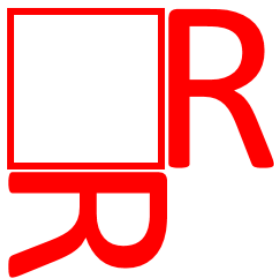
Alternatively, since we know that  $R^2$  is also correlation squared with SLR models, we could just as easily use the  $R^2$ 's from the two collinearity regression above to measure the extent to which the two variables, wk1 and wk2, moved together in a linear fashion. Remember that  $-1 \leq \rho \leq 1$  and  $0 \leq \rho_{xy}^2 = R^2 \leq 1 \dots$  so  $R^2$  reflects the magnitude of the correlation, but not the sign.

Note that the R-sq's in the two collinearity regressions above are the same, .873. But this should be no surprise, since you know that  $R^2$  is also correlation squared, and indeed,  $.9342^2 = .873$ .

So for the case of just two RHS variables, we have two equivalent ways of measuring collinearity:

- **Correlation -  $\rho_{wk_1, wk_2}$**  : Use the simple correlation between the two RHS variables, or
- **Coefficient of Determination -  $R^2$**  : Use the  $R^2$  from the collinearity regression (in which one of the RHS variables is regressed on the other).

Both approaches will give you effectively the same metric. But they differ in one very important respect: By definition, the correlation concept can be applied only to pairs of explanatory variables. In contrast, the  $R^2$  from the collinearity regression approach **can be easily extended** to MLR models with more than just two RHS variables.



And so that's what we do to measure multicollinearity.... we employ  $R^2$  from the collinearity regression.

*Measuring multicollinearity:* Use the  $R^2$  from the collinearity regression.

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... with Three or more RHS Variables: *R-squared*<sub>*j*</sub> ( $R_j^2$ )

Since the correlation concept applies only to pairs of variables, it will not be useful in assessing collinearity when we have three or more RHS variables. But as just discussed, the R-squared metric easily extends to MLR models with additional RHS variables, and so we will use that metric to measure multicollinearity.

An example will show you how this is done.

**An Example:** Now consider a simple box office revenue forecasting MLR model in which total gross revenue, *rtotgross*, is regressed on weekly revenues in weeks #1, #2 and #3. Here's a summary of the results from the full MLR model (Model (1)), and the three associated collinearity regressions (Models (2), (3) and (4)). The column headers tell you the dependant variable in each model.

	<u>MLR Model</u>	<u>Collinearity Regressions</u>		
	(1)	(2)	(3)	(4)
	<i>rtotgross</i>	<i>wk1</i>	<i>wk2</i>	<i>wk3</i>
<i>wk1</i>	0.540*** (21.36)		0.261*** (111.42)	-0.115*** (-34.25)
<i>wk2</i>	0.745*** (9.79)	2.361*** (111.42)		0.792*** (131.20)
<i>wk3</i>	4.778*** (59.84)	-1.146*** (-34.25)	0.872*** (131.20)	
<i>_cons</i>	-0.601** (-2.64)	0.110 (1.07)	0.0817* (2.40)	0.287*** (8.91)
N	7730	7730	7730	7730
<b>R-sq</b>	<b>0.921</b>	<b>0.886</b>	<b>0.959</b>	<b>0.908</b>

t statistics in parentheses  
\* p<0.05, \*\* p<0.01, \*\*\* p<0.001

As suggested above, the R-sq's now differ across the three collinearity regressions:

- *wk2* is the most collinear explanatory variable since the  $R^2$  in Model (3) is .959 (which tells us that 95.9% of the variation in the *wk2* variable can be explained by a linear function of the other two explanatory variables, *wk1* and *wk3*),
- *wk3* in model (4) is less collinear, with  $R^2 = .908$ , and
- *wk1* is the least collinear of the explanatory variables in Model (1), with  $R^2 = .886$ .

We call the R-sq's in the collinearity regressions the *R-squared j* measures of collinearity,  $R_j^2$ , where the j index tells you which RHS variable,  $x_j$ , is the dependant variable in the collinearity regression.



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- If  $R_j^2$  is close to 1, then most of the variation in the RHS variable  $x_j$  can be explained by the other explanatory variables in the model. And so in that sense  $x_j$  doesn't bring much explanatory power that's new to the model (or offer much new independent explanatory power). In this case we say that  $x_j$  is highly collinear with the other RHS variables.
- But if  $R_j^2$  is small, then  $x_j$  is not so collinear with the other explanatory variables, and in that sense it potentially brings a lot of new explanatory power to the RHS of the model. In this case we say that  $x_j$  is not so collinear with the other RHS variables.

### Variance Inflation Factors (VIFs): *Easily generate $R_j^2$ 's*

Running all those collinearity regressions can be a nuisance. Fortunately, regression packages are often able to generate *Variance Inflation Factors (VIFs)*, which provide a quick and simple path to the  $R_j^2$ 's. You may be wondering what any of this has to do with *Variance Inflation*.

That will become quite clear when we later turn to MLR inference, and the variance of the parameter estimators.

$$\text{VIFs are defined by: } VIF_j = \frac{1}{1 - R_j^2}, \text{ so that } R_j^2 = 1 - \frac{1}{VIF_j}.$$

# VIF

Accordingly, if you know the VIFs, it's easy to calculate the  $R_j^2$ 's ... and *vice-versa*.

**Example:** Here's an example using the previous MLR analysis:

Source	SS	df	MS	Number of obs	=	7,730
Model	27234043.1	3	9078014.37	F(3, 7726)	=	30052.55
Residual	2333803.56	7,726	302.07139	Prob > F	=	0.0000
				R-squared	=	0.9211
				Adj R-squared	=	0.9210
Total	29567846.7	7,729	3825.57209	Root MSE	=	17.38

rtotgross	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
wk1	.5403215	.0253012	21.36	0.000	.4907242 .5899187
wk2	.7447428	.0760905	9.79	0.000	.5955848 .8939008
wk3	4.77794	.0798398	59.84	0.000	4.621432 4.934447
_cons	-.6009747	.2274986	-2.64	0.008	-1.046934 -.1550159

. vif

Variable	VIF	1/VIF	Rsq j
wk2	24.62	<b>0.040613</b>	<b>0.959387</b>
wk3	10.88	<b>0.091905</b>	<b>0.908095</b>
wk1	8.79	<b>0.113804</b>	<b>0.886196</b>
Mean VIF	14.76		

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And to generate the  $R_j^2$ 's, we just exploit their relationship with the VIFs:  $R_j^2 = 1 - \frac{1}{VIF_j}$ , and so

$$R_2^2 = 1 - .0406 = .9594, R_3^2 = 1 - .0919 = .9081 \text{ and } R_1^2 = 1 - .1138 = .8862.$$

These results agree with the earlier  $R_j^2$ 's results from the collinearity regressions.

### Endogeneity (Omitted Variable Bias/Impact)

As discussed previously, estimated coefficients will be *biased* (or less pejoratively, *impacted*) to the extent that those variables are correlated with omitted variables, which are themselves correlated with the dependent variable.

**Misleading?** And as also discussed earlier, this is not so much a *bias* as a matter of *interpretation*. The estimated coefficients reflect the average incremental relationship between changes in the particular RHS variable and changes in the LHS variable, controlling for all the other RHS variables in the model. But of course, if a RHS variable is omitted/dropped/excluded from the model, it's not the same model... and so no one should be surprised to see changes in the OLS/MLR coefficient estimates for the surviving variables.

### Endogeneity Case I: $k=2$ to $k=1$

*Starting with two RHS variables, and dropping one.*

Lets return to the box office revenue analysis in the previous section, and an initial MLR model in which *rtotgross* is regressed on weekly revenues *wk1* and *wk2*. As in the earlier section, we will focus on what happens to the estimated *wk1* coefficient when *wk2* is dropped from the analysis.

You will see that the Omitted Variable Bias/Impact (OVB) on the estimated *wk1* coefficient is the product of:

- 1) **Collinearity Regression (SLR)**: the estimated *wk1* coefficient in the collinearity regression of the omitted variable, *wk2*, on the surviving variable, *wk1*, and
- 2) **Full Model (MLR)**: the estimated *wk2* coefficient in the full model.

And the sign of the OVB will accordingly depend on the signs of those two coefficients.

- 1) **Collinearity Regression (SLR)**: Since the collinearity regression is a SLR model, the sign of the estimated slope coefficient in that model is the same as the sign of the correlation between *wk1* and *wk2* revenues.
- 2) **Full Model (MLR)**: And we tend to believe that the signs of the coefficients in the full model reflect the correlations between those RHS variables and the dependent variable. As you'll see later, that is not quite correct. But it's not a bad way to think about things... and close to accurate if the two RHS variables are fairly independent.

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The OLS results:

- Model (1) is the full model in which *rtotgross* has been regressed on *wk1* and *wk2*.
- Model (2) shows the results of the SLR model after *wk2* was dropped/omitted from Model (1). In going from Models (1) to (2), the estimated *wk1* coefficient increases from -0.0120 to 2.354, an increase of 2.366.
- And Model (3) presents the results of the collinearity regression in which *wk2*, the variable omitted/dropped in Model (2) is regressed on *wk1*, the surviving/remaining explanatory variable in Model (2).

	<u>Full Model</u> (1)	<u>wk2 dropped</u> (2)	<u>Collinearity Reg</u> (3)
	<i>rtotgross</i>	<i>rtotgross</i>	<i>wk2</i>
<i>wk1</i>	-0.0120 (-0.46)	2.354** (176.20)	0.522** (249.91)
<i>wk2</i>	4.536** (95.87)		
<i>_cons</i>	0.401 (1.74)	4.433*** (13.83)	0.889*** (17.75)
N	9114	9114	9114
R-sq	0.887	0.773	0.873

t statistics in parentheses  
\* p<0.05, \*\* p<0.01, \*\*\* p<0.001

OVB impact on *wk1* coefficient when *wk2* dropped from the model:

- The change in the *wk1* coeff.:  $2.354 - (-0.012) = 2.366 \dots$
- The OVB calc:  $\{\textit{wk2} \text{ coeff in (1)}\} \{\textit{wk1} \text{ coeff in (3)}\} = 4.536 \cdot 0.522 = 2.366$

Notice that since both of the components in the calculation are positive, OVB is positive.

The example above is contrived in the sense that we have *wk2* data, and so there's really no need to drop it from the model. Nonetheless, the example is illustrative, and usefully highlights the two factors that drive OVB:

- ***wk1* coeff. in SLR Collinearity Model:** the estimated coefficient for the surviving variable (*wk1*) in the collinearity regression when the omitted variable (*wk2*) is regressed on the surviving variable (*wk1*), and
- ***wk2* coeff. in the MLR Full Model:** the estimated coefficient for the omitted variable (*wk2*) when it's in the full model (1)

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**More generally (dropping z):** Suppose that you start with a MLR model in which  $y$  is regressed on two RHS variables,  $x$  and  $z$ . You are interested in estimating the change in the estimated  $x$  coefficient if  $z$  is dropped from the model. The following table summarizes the qualitative effects of omitted variable bias when you omit, say,  $z$ , from the model and just regress  $y$  on  $x$ .

To fix notation, here are the SRFs for the full model and the collinearity regression model of interest (the SRF subscripts reference the dependent variable in each model):

- **Full Model - SRF<sub>y</sub>:**  $\hat{y} = \hat{\beta}_0 + \hat{\beta}_x x + \hat{\beta}_z z$
- **Collinearity Regression - SRF<sub>z</sub>:**  $\hat{z} = \hat{\alpha}_0 + \hat{\alpha}_x x$  (the omitted variable,  $z$ , is regressed on the surviving variable,  $x$ )

**Omitted Variable Bias** (dropping  $z$ ; impact on the  $x$  coeff.:  $\hat{\alpha}_x \hat{\beta}_z$ )

		$\hat{\beta}_z$ from the <b>MLR Full Model</b> (SRF <sub>y</sub> )		
$\hat{\alpha}_x$ from the <b>SLR Collinearity Regression</b> (SRF <sub>z</sub> )		$\hat{\beta}_z > 0$	$\hat{\beta}_z = 0$	$\hat{\beta}_z < 0$
$\hat{\alpha}_x > 0$		positive	0	negative
$\hat{\alpha}_x = 0$		0	0	0
$\hat{\alpha}_x < 0$		negative	0	positive

**Signing the impact/bias:** This table is useful because often, and especially with *favorite coefficient* models, you want to be able to sign the bias if you have run out of fixes... so that even if you don't know the actual magnitude, you can still say whether you have under- or over-estimated the  $x$  parameter in the full model (as discussed earlier).

And when thinking about the signs of these coefficients, we often look at correlations:

- Since the collinearity regression is an SLR model, the sign of  $\hat{\alpha}_x$  will reflect the sample correlation of  $x$  and  $z$ . So in this case, that sign of the  $x$ - $z$  correlation is informative.
- While it's often the case that the sign of  $\hat{\beta}_z$  will reflect the sample correlation of  $z$  with the dependent variable  $y$ , it turns out that that need not always be the case. And so in this case, the correlation might not be as informative as we might like. We will return to this issue later when we later introduce the concept of *partial* correlations.



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But let's hope that you do not have to resort to the hypothetical exercise of signing the omitted variable impact/bias. As mentioned earlier, there are more concrete ways of attacking the problem:

- Don't be lazy; find the data!... and if that doesn't work...
- Perhaps use a proxy variable for the omitted variable... and if that doesn't work ... and only as a last resort...
- Turn to Instrumental Variables (IVs) (discussed later in the course).

**#1: Data Integrity**

**#2: Endogeneity**

**#3: ... not close!**

I can't say it enough: After data integrity, endogeneity (omitted variable impact/bias) is the most important issue in doing econometrics. What variables did you leave out of your analysis? ... and how are those omissions biasing/impacting your coefficient estimates? Inquisitive minds want to know!

### Endogeneity Case II: $k > 2$ to $k-1$ *What's the diff?*

*Starting with more than two RHS variables, and dropping one explanatory variable.*

In looking at endogeneity so far, we've focused on going from two explanatory variables to one. How do things change if we have more RHS variables in the full model? *Not much! Really!*

I will illustrate with an example, which starts with three RHS variables ( $x$ ,  $z$  and  $w$ ), and excludes  $w$  from the analysis, leaving the two surviving RHS variables  $x$  and  $z$ .<sup>3</sup> To assess the impact of OVB, you are focused on how the estimated  $x$  and  $z$  coefficients (for the surviving variables) are impacted by omitting  $w$  from the full model. Unlike the previous OVB analysis, we now have omitted variable impacts on multiple surviving explanatory variables.

As before, the impacts on those two coefficients (the omitted variable bias/impact) will be driven by coefficients in two models: 1) the **Full Model**, and 2) the **Collinearity Regression**.

**Full Model:** This model includes the third explanatory variable,  $w$ :

- **Full Model - SRF<sub>y</sub>:**  $\hat{y} = \hat{\beta}_0 + \hat{\beta}_x x + \hat{\beta}_z z + \hat{\beta}_w w$

**Collinearity Regression:** As before, you also run the collinearity regression, regressing the omitted variable,  $w$ , on the two surviving/remaining variables in the model,  $x$  and  $z$ :

- **Collinearity Regression - SRF<sub>w</sub>:**  $\hat{w} = \hat{\alpha}_0 + \hat{\alpha}_x x + \hat{\alpha}_z z$

Then the omitted variable biases/impacts from excluding  $w$  from the model are the same sorts of products of coefficients that you saw before:

$$OVB_x = \hat{\alpha}_x \hat{\beta}_w \text{ (the product of the SRF}_w \text{ } x \text{ coeff and the SRF}_y \text{ } w \text{ coeff)}$$

$$OVB_z = \hat{\alpha}_z \hat{\beta}_w \text{ (the product of the SRF}_w \text{ } z \text{ coeff and the SRF}_y \text{ } w \text{ coeff)}$$

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<sup>3</sup> The analysis with more than three RHS variables in the Full Model would be the same... so we'll stay with just three explanatory/RHS variables.

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**Comparing OVB magnitudes:** Since  $\hat{\beta}_w$  (the  $w$  coefficient in the Full Model) is common to both calculations, differences in OVBs are driven entirely by the different coefficients in the Collinearity Regression. And so for example, the variable with the largest (in magnitude) coefficient in the Collinearity Regression, will also be the variable experiencing the largest OVB impact.

And so the methodology for computing the omitted variable impact is as before, except now there are two explanatory variables in the collinearity regression... and two impacts to be calculated.

**Example:** Here's an example, returning to the *movierevs* dataset: Model (1) is the *Full Model*; *wk3* has been dropped for Model (2); Model (3) is the collinearity regression in which the omitted variable *wk3* has been regressed on the two surviving variables, *wk1* and *wk2*.

	<u>Full Model</u> (1)	<u>wk3 dropped</u> (2)	<u>Collinearity Reg</u> (3)
	<u>rtotgross</u>	<u>rtotgross</u>	<u>wk3</u>
wk1	0.540 (21.36)	-0.00941 (-0.33)	-0.115*** (-34.25)
wk2	0.745* (9.79)	4.528*** (88.37)	0.792*** (131.20)
wk3	4.778*** (59.84)		
_cons	-0.601** (-2.64)	0.772** (2.82)	0.287*** (8.91)
N	7730	7730	7730
R-sq	0.921	0.884	0.908

t statistics in parentheses  
\* p<0.05, \*\* p<0.01, \*\*\* p<0.001

Notice that when *wk3* was dropped from Model (1):

- the estimated *wk1* coefficient decreased by 0.549, from 0.540 to -0.00941, and
- the estimated *wk2* coefficient increased by 3.783, from 0.745 to 4.528.

Applying the formulas above, we estimate the omitted variable biases using the product of the *wk3* coefficient in Model (1), 4.778, and the respective RHS variable coefficients in the collinearity regression, Model (3):

- *wk1* OVB:  $4.778 * (-0.115) = -.549$ , as advertised
- *wk2* OVB:  $4.778 * (0.792) = 3.784$ , almost as advertised... blame rounding error

Notice that as discussed above, since in both cases the collinearity coefficients are being multiplied by 4.778, the *wk3* coefficient in Model (1), the magnitudes of the OVB's will be driven entirely by differences in the coefficients in Model (3). And so by inspection, since the *wk2* coefficient in Model (3), 0.79, is about seven times the magnitude of the *wk1* coefficient, 0.11, you should expect that the OVB for *wk2* will be about 7x the OVB magnitude for *wk1*,

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which is the case. So the collinearity regression coefficients alone provide you with a sense of the relative magnitudes of the OVB impacts when dropping  $wk3$  from the model.

**Case III:  $k > 2$  to  $k - ?$**  - *Starting with more than two RHS variables, and dropping more than one.*

Unfortunately, the analysis of omitted variable impact gets much much more complicated when multiple explanatory variables have been omitted from the analysis. That's a shame, of course, as the typical case with MLR models is that you have indeed omitted multiple explanatory factors... and it sure would be nice to know how that has impacted your estimated coefficients.

So we have some intuition from the specific cases that we've analyzed...but have clearly fallen short of expectations!

### Interpreting Coefficients II: *What's New? ... and What's Left?*

Earlier you saw an SRF-based interpretation of estimated slope coefficients in MLR models, in which estimated coefficients provide predicted average incremental effects, *ceteris paribus*. And so, for example, if *rtotgross* has been regressed on *wk1* and *wk2* (weekly gross revenues), then the estimated *wk1* coefficient captures the average effect on predicted total revenues associated with changes in *wk1* revenues, holding *wk2* revenues fixed.

We now turn to a second very different interpretation of the estimated MLR coefficients, which will focus instead on *What's New* and *What's Left*:



- ***What's New*** (independent contribution) about individual explanatory/RHS variables, given the other RHS variables in the analysis, and
- ***What's Left*** (still to be explained) in the dependent/y variable, given that you've already included the other RHS variables in the analysis.

**To fix terms:** Consider a OLS/MLR model in which the dependent variable  $y$  has been regressed on a bunch of RHS variables  $x_1, x_2, x_3 \dots$  and let  $\hat{\beta}_1$  be the estimated  $x_1$  coefficient in the MLR model. Our goal is to better understand  $\hat{\beta}_1$  as capturing the relationship between *What's New* about  $x_1$  and  $y$ , or perhaps instead, *What's Left* with  $y$ . There will be two approaches here, each of which requires that we first determine *What's New* about  $x_1$ .

***What's New* about  $x_1$ :**

1. Run the collinearity regression for RHS variable  $x_1 \dots$  so regress  $x_1$  on the other RHS variables.
2. Capture the *residuals* from the collinearity regression.

The *predicted*s from the collinearity regression generate the part of  $x_1$  *explained* by the other RHS variables. And so the *residuals* capture the part of  $x_1$  *unexplained* by the

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other RHS variables. In a sense, those residuals capture the part of  $x_1$  that is new to the RHS of the model (given the other RHS variables in the analysis).

We say that those residuals are *WhatsNew* about  $x_1$ , given the other RHS variables in the model.

### Interpretation II.a: Correlation between $y$ and *What's New* about $x_1$

$\hat{\beta}_1$  reflects the correlation between the dependent variable  $y$  and *What's New* about the RHS variable  $x_1$ . If you run a SLR model regressing  $y$  on *WhatsNew* about  $x_1$ , you'll discover that the estimated slope coefficient in that SLR model is the same as  $\hat{\beta}_1$ , the MLR coefficient!

And since SLR coefficients essentially reflect correlations (subject to a standard deviations adjustment), it's useful to interpret MLR coefficients as capturing the correlation between the dependent variable,  $y$ , and *WhatsNew* about an explanatory variable. Or to be more precise, the sign of the estimated MLR coefficient will agree with the sign of the correlation between the dependent variable and *WhatsNew* about that variable.

**Example:** Here's an example... working with *rtotgross* and *wk1* and *wk2* revenues (we'll focus on the *wk2* coefficient).

1. Run the baseline MLR Model (1): `reg rtotgross wk1 wk2`
2. Run the Collinearity regression, Model (2): `reg wk2 wk1`
3. Create *whatsnew2* by capturing the residuals: `predict whatsnew2, resid`
4. Run the (equivalent) SLR model, Model (3): `reg rtotgross whatsnew2`

	(1)	(2)	(3)
	rtotgross	wk2	rtotgross
wk1	-0.0120 (-0.46)	0.522*** (249.91)	
<u>wk2</u>		<u>4.536***</u> (95.87)	
<u>whatsnew2</u>			<u>4.536***</u> (34.23)
_cons	0.401 (1.74)	0.889*** (17.75)	28.64*** (50.05)
N	9114	9114	9114
R-sq	0.887	0.873	0.114
rss (SSR)	3,466,910.0	169,974.8	27,194,258.1
rmse (RMSE)	19.51	4.319	54.63

t statistics in parentheses  
 \* p<0.05, \*\* p<0.01, \*\*\* p<0.001

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As predicted, the estimated MLR Model (1) *wk2* coefficient is the same as the SLR Model (3) slope coefficient.

Accordingly, every estimated MLR coefficient can be computed using a SLR model in which the dependent variable is regressed on *What's New* about each particular explanatory variable. And so the signs of the estimated MLR coefficients will agree with the signs of the correlations of the dependent variable with *What's New* about each of the RHS variables.

Note that in the previous example, while the *wk2* coefficients in the Models (1) and (3) agree, the other reported regression statistics/figures (R-sqs, SSRs, RMSEs, t stats, etc.) are wildly different. So those two models are not at all comparable, even though they do agree on the estimated coefficients of interest. The second interpretation of  $\hat{\beta}_1$  involving *What's New* about  $x_1$  will instead feature models with quite similar results, across the board.

**Interpretation II.b:** Correlation between *What's Left* of  $y$  and *What's New* about  $x_1$

$\hat{\beta}_1$  also reflects the correlation between *What's Left* of the dependent variable  $y$  and *What's New* about the RHS variable  $x_1$ .

As just suggested above, If we bring *WhatsLeft* into the analysis, we'll have a SLR model that much more closely resembles the original MLR model, slope coefficients and all. ***Let's do that!***

### *... and WhatsLeft*

The *WhatsLeft* – *WhatsNew* interpretation (II.b) continues the previous build:

1. Return to the previous OLS/MLR model in which the dependent variable  $y$  has been regressed on a bunch of RHS variables  $x_1, x_2, x_3 \dots$  and as before, let  $\hat{\beta}_1$  be the estimated  $x_1$  coefficient in the MLR model. As above, generate *WhatsNew* about  $x_1$  by regressing  $x_1$  on the other RHS variables and capturing the residuals.

2. To generate *WhatsLeft* about  $y$ , regress  $y$  on the *other* RHS variables (on the RHS variables other than  $x_1$ ).

**What's left?**

3. Capture the *residuals* from this regression.

The *predicted*s from the regression generate the part of  $y$  *explained* by the other RHS variables... and so the *residuals* capture the part of  $y$  *unexplained* by the other RHS variables. We'll call that unexplained part *WhatsLeft* (about  $y$ ).

4. Run a SLR model regressing *WhatsLeft* (about  $y$ ) on *WhatsNew* (about  $x_1$ ).

As before, the estimated coefficient in that SLR model will be equal to  $\hat{\beta}_1$ , the MLR coefficient! But in contrast to the previous SLR model, the performance metrics for the new model will look a lot like those in the MLR model. In many cases, they won't be precisely the same figures (usually due to small differences in degrees of freedom (*dofs*))... but they will be very very close.

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To illustrate, let's continue working with the *rtotgross*, *wk1* and *wk2* box office revenues MLR model: Generate *WhatsNew* as before.

And to generate *WhatsLeft*:

5. Regress *rtotgross* on *wk1*: `reg rtotgross wk1`
6. Create *whatsleft1* by capturing the residuals: `predict whatsleft1, resid`
7. Run the (equivalent) SLR model: `reg whatsleft1 whatsnew2`

If you compare the estimated MLR and SLR slope coefficients, you'll find that they agree! And if you also compare the MLR and SLR SSRs, MSEs, R-sq's, t stats etc... you'll discover that they pretty much agree, except for SSTs and R-sq's.

The following example illustrates:

**Example:** Here are the results from the original model and the regression of *WhatsLeft* on *WhatsNew*:

**Model (1):** `reg rtotgross wk1 wk2`

Source	SS	df	MS	Number of obs	=	9,114
Model	27224699.6	2	13612349.8	F(2, 9111)	=	35773.10
Residual	<b>3,466,910</b>	<b>9,111</b>	<b>380.519153</b>	Prob > F	=	0.0000
				<b>R-squared</b>	=	<b>0.8870</b>
				Adj R-squared	=	0.8870
Total	30691609.6	9,113	3367.89308	<b>Root MSE</b>	=	<b>19.507</b>

rtotgross	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
wk1	-.0120343	.0264237	-0.46	0.649	-.0638307	.039762
<b>wk2</b>	<b>4.536046</b>	<b>.0473147</b>	<b>95.87</b>	<b>0.000</b>	<b>4.443298</b>	<b>4.628793</b>
_cons	.4006355	.2300356	1.74	0.082	-.050286	.8515569

**Model (2):** `reg whatsleft1 whatsnew2`

Source	SS	df	MS	Number of obs	=	9,114
Model	3497351.55	1	3497351.55	F(1, 9112)	=	9192.01
Residual	<b>3,466,910</b>	<b>9,112</b>	<b>380.477398</b>	Prob > F	=	0.0000
				<b>R-squared</b>	=	<b>0.5022</b>
				Adj R-squared	=	0.5021
Total	6964261.6	9,113	764.211741	<b>Root MSE</b>	=	<b>19.506</b>

whatsleft1	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
<b>whatsnew2</b>	<b>4.536046</b>	<b>.0473121</b>	<b>95.87</b>	<b>0.000</b>	<b>4.443303</b>	<b>4.628788</b>
_cons	1.23e-09	.2043195	0.00	1.000	-.4005122	.4005122

Some comments:

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- As promised, the *wk2* slope coefficients are the same in the previous two models, as are the SSRs. And so subject to an adjustment for the standard deviations, the MLR slope coefficients for any RHS variable reflect the **correlations** between *WhatsNew* about that variable and *WhatsLeft* in the *y* variable (given the rest of the model).
- The *dofs* are slightly different (9,111 v. 9,112), reflecting the difference between the MLR and SLR models. And the *MSEs* (*RMSEs*) are very very close, but slightly different because the *dofs* are slightly different.
- We won't get to them for a bit (when we discuss statistical inference), but notice that the other reported *wk2* figures (Std. Err., t stat,  $P > |t|$  and [95% Conf. Interval]) are also virtually identical in the two models... but again, they are not exactly the same due to the different *dofs* in the two models.
- The *R-sq's* are quite different... but only because the *SSTs* are so different (not surprising given that *WhatsLeft* captures the variation in *rtotgross* not explained by *wk1*).

### **Important Take Away:**

Earlier, in the initial discussion of endogeneity, I said:

*...we tend to believe that the signs of the coefficients in the full [MLR] model reflect the correlations between those RHS variables and the dependent variable. As you'll see later, that is not quite correct. But it's not a bad way to think about things... and close to accurate if the two RHS variables are fairly independent.*

And now you see why!

Coefficients in MLR models **do not** necessarily reflect simple pairwise correlations between the dependent variable and the RHS variables. But they **do** reflect:

- correlations between the dependent variable and *What's New* about RHS variables, as well as
- correlations between *What's Left* of the dependent variable and *What's New* about RHS variables. (You will learn later that these are called *partial correlations*.)

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### *And just to drive the point home....*

Here are two examples, in which the signs of the simple pairwise correlations do not agree with the signs of the respective coefficients in the MLR models.

	(1) Brozek	(2) Brozek		(1) rtotgross	(2) rtotgross
wgt	0.162*** (12.27)	-0.136*** (-7.08)	wk1	2.354*** (176.20)	-0.0120 (-0.46)
abd		0.915*** (17.42)	wk2		4.536*** (95.87)
_cons	-9.995*** (-4.18)	-41.35*** (-17.14)	_cons	4.433*** (13.83)	0.401 (1.74)
N	252	252	N	9114	9114

Note that the signs of the slope coefficients in the SLR models will agree with the signs of the pairwise correlations. Blame multicollinearity!

- **Box Office Revenues: *wk1* and *wk2***

*wk1* revenues are positively correlated with total box office revenues but *wk1* has a negative coefficient in the MLR model that includes *wk2* (left).

- **Bodyfat: *wgt* and *abd***

*wgt* is positively correlated with the *Brozek* measure of bodyfat ( so *wgt* has a positive slope coefficient in the SLR model), but *wgt* has a negative coefficient in the MLR model that includes *abd* (waist size) (right).

So while the signs of correlations between RHS variables and the dependent variable often agree with coefficient signs in MLR models, that is certainly not guaranteed. All of this, of course, makes signing omitted variable bias all the more challenging. Figuring out the signs of partial correlations is hard work! ... and never ever fall into the trap of just thinking about pairwise correlations (unless, of course, your RHS variables are all independent of one another).

We'll return to this topic.